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Viscous–gravity spreading of an oil slick

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A two-dimensional oil slick is examined as it spreads on water under the influence of gravitational and viscous forces. An exact analytical and numerical description of the flow is obtained in a certain rational asymptotic limit. The central result is an expression for the size of the slick as a function of time. This expression contains no free parameters to be empirically determined, and is in substantial agreement with experiment.

1. Introduction

In an ecologically minded age, it is not surprising that there is substantial interest in understanding and describing the physical processes that govern the spread of an oil slick on water. A recent review article (Hoult 1972) makes it quite clear that, at the present time, we have a crude qualitative understanding of the mechanisms involved, but much remains to be done in providing a detailed quantitative picture of the flow.

Fay (1969) has shown, by essentially dimensional reasoning, that there are three stages in the spread of an initially concentrated volume of oil. At first, there is a balance between gravitational forces, which tend to spread the oil over the surface, and resistive inertial forces which arise from the associated acceleration. After some time has elapsed, the slick becomes so thin that the inertial forces become negligible in comparison with the viscous drag generated by the flow of the oil over the water. At even larger times, the third stage emerges, in which surface tension replaces gravity as the driving force.

These three stages are all characterized by power laws governing the size of the slick as a function of time. The exponents in these laws are well confirmed by experiment, for both plane and axisymmetric slicks (Hoult 1972). However, Fay's analysis does not predict the proportionality constants in these laws, nor does it provide any details of the distribution of velocity in the flows.

A start on these more complicated questions has been advanced by Hoult (1972), who suggests that associated with each of the last two stages are simple similarity solutions of the governing equations. There are several reasons to believe that this idea contains flaws, and some of them will be discussed in § 2. Nevertheless, Hoult's discussion contains several ideas that are crucial in generating a rational analytic theory, and the present author's indebtedness to this work will soon become apparent.

The problem examined in this paper is that of a two-dimensional slick in the second stage, that is, a slick spreading under the influence of gravitational and viscous forces. The primary concern is to derive, with no *ad hoc* assumptions, an explicit expression for the size of the slick as a function of time. We shall start by repeating those elements of Fay's discussion that are relevant to the present problem, as well as listing the basic assumptions that are needed to carry out the analysis.

Suppose that the extent of the slick is characterized by a length l and its thickness by h. Then the inertial force acting on the oil has order of magnitude

$$\rho l t^{-2}(hl).$$

Here, ρ is the density of the oil, but this is assumed to be so close to the water density in value that it is only necessary to distinguish between them in calculating the net gravitational force. Time is represented by t.

The water boundary layer has thickness of order $(\nu t)^{\frac{1}{2}}$, where ν is the kinematic viscosity. Consequently the viscous drag acting on the slick is

$$\rho \nu l t^{-1}(\nu t)^{-\frac{1}{2}} l.$$

Comparing the above two retarding forces, we see that the viscous drag will dominate the d'Alembert force provided

$$(\nu t)^{\frac{1}{2}} \gg h, \tag{1.1}$$

that is, if the boundary-layer thickness is much larger than the thickness of the slick. This is an important simplification in that, as far as the boundary-layer calculations are concerned, the slick may be regarded as a flat sheet.

Of course, a boundary-layer description of the water flow is only appropriate if the Reynolds number is large, which implies that

$$l \gg (\nu t)^{\frac{1}{2}}. \tag{1.2}$$

The gravitational driving force has magnitude

$$\rho g \Delta h h,$$

where Δ is the fractional density difference between the oil and the water. This balances the viscous drag provided

$$l \sim t^{\frac{3}{8}}(g\Delta)^{\frac{1}{4}}\, V^{\frac{1}{2}} \nu^{-\frac{1}{8}}, \tag{1.3}$$

where $V \sim hl$ is the slick volume. This is one of Fay's results, and the exponent $\frac{3}{8}$ has been verified experimentally.

The inequalities (1.1) and (1.2) can now be rewritten as

$$(g\Delta)^2\, V^4 \nu^{-5} \gg t \gg (g\Delta)^{-\frac{2}{7}}\, V^{\frac{4}{7}} \nu^{-\frac{3}{7}}.$$

Thus the theory we shall develop is valid for large times, but not *too* large in a sense that depends essentially on the volume of the slick, the only variable parameter once the liquids are specified.

Other assumptions are also needed in the course of the analysis. Thus the net surface tension force is assumed to be so small that it can be neglected. In addition, considerable simplification arises if the oil velocity does not change significantly across the thickness of the slick. Such a slug flow description is valid provided

$$\rho \nu h \mu_0^{-1}(\nu t)^{-\frac{1}{2}} \ll 1,$$

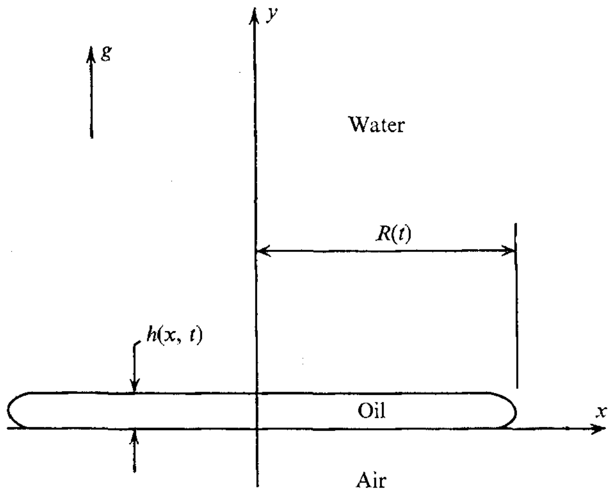

FIGURE 1. Oil slick on water.

where μ_0 is the oil viscosity. This may be rewritten as

$$(\rho\nu/\mu_0)^{\frac{8}{7}} \ll t(g\Delta)^{\frac{2}{7}} V^{-\frac{4}{7}}\nu^{\frac{3}{7}}. \tag{1.4}$$

Hoult (1972) advocates use of the slug flow approximation, which he claims is valid provided the oil viscosity is large enough. However, the right side of the above inequality is much larger than 1, by hypothesis, so that the assumption is accurate without any undue restrictions on the oil viscosity.

Finally, since the slick and the boundary layer are thin, there is hydrostatic equilibrium in the vertical direction.

2. Analysis of the flow field

The oil slick is assumed to have width $2R(t)$, where

$$R(t) = Ct^{\alpha}. \tag{2.1}$$

α is known to have the value $\frac{3}{8}$ (equation (1.3)), but this should also emerge from our analysis. The dependence of C on the various flow parameters is also known from (1.3), but the actual numerical value is not known *a priori*. The basic purpose of our analysis is to derive an explicit expression for C.

The slick lies in $y = 0$ (figure 1), with the water occupying the half-space $y > 0$. The half-space $y < 0$ is occupied by air, which plays no role in the analysis.

The equations governing the water flow are the unsteady boundary-layer equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{2.2}$$

$$\partial u/\partial x + \partial v/\partial y = 0. \tag{2.3}$$

The oil slick is characterized by a velocity $q(x, t)$ and its thickness $h(x, t)$. Thus the continuity equation for the oil is

$$\partial h/\partial t + \partial(qh)/\partial x = 0. \tag{2.4}$$

The gravitational forces generate a horizontal pressure gradient in the oil, and the force associated with this is balanced by the shear stress acting at the oil–water interface. Inertial forces play no role in this balance by virtue of (1.1). Thus the momentum equation for the oil may be written (Hoult 1972) as

$$-\frac{\nu}{g\Delta}\frac{\partial u}{\partial y}(x,0;t)+h\frac{\partial h}{\partial x}=0. \tag{2.5}$$

The mathematical formulation is completed by certain boundary conditions. Thus,

$$\left.\begin{aligned} v(x,0;t)&=0\\ u(x,0;t)&=q(x,t)\end{aligned}\right\}\quad \text{in}\quad |x|<R(t) \tag{2.6a}$$

and

$$\lim_{y\to\infty}u(x,y;t)=0. \tag{2.6b}$$

In addition there is symmetry about the y axis, and at the edge of the slick q is equal to the edge velocity, i.e.

$$q(R(t),t)=\dot{R}(t). \tag{2.7}$$

The following argument suggests that the problem is properly posed, at the same time describing the basic details of the procedure used to solve it.

Suppose that the velocity q is a given function of x and t. Then the boundary-layer problem in the water can be solved, in principle, giving rise to a known distribution of skin friction. Equation (2.5) can then be integrated to determine $h(x,t)$, noting that h vanishes at the edge of the slick. The continuity equation (2.4) will then only be satisfied if the original choice of q is correct. Once a self-consistent solution has been obtained, the volume of the slick can be calculated by integrating h. This determines the unknown constant C in (2.1).

At first sight this might seem to be a very difficult problem, since, for example, the boundary-layer flow apparently depends on three independent variables. However, although the problem does have a reference length, namely $V^{\frac{1}{2}}$, and also a reference velocity $(g\Delta)^{\frac{1}{2}}V^{\frac{1}{4}}$, the relation (2.1) suggests that these only appear in the particular combination C. In that case the flow depends only on two independent variables, and this is the essential foundation of our analysis.

2.1. *Structure in the neighbourhood of the origin*

It is instructive to consider the nature of the boundary-layer flow in the neighbourhood of $x=0$. To this end we introduce the stream function defined by

$$u=\partial\psi/\partial y,\quad v=-\partial\psi/\partial x,$$

which satisfies the equation

$$\frac{\partial^2\psi}{\partial y\,\partial t}+\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\,\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\nu\frac{\partial^3\psi}{\partial y^3}. \tag{2.8}$$

A solution is sought in the form

$$\psi=\nu^{\frac{1}{2}}xt^{-\frac{1}{2}}F(\gamma,\beta), \tag{2.9}$$

where

$$\gamma=x/Ct^{\alpha},\quad \beta=y/(\nu t)^{\frac{1}{2}}.$$

The function F then satisfies the equation

$$F_{\beta\beta\beta}+F_{\beta\beta}(\tfrac{1}{2}\beta+F)+F_{\beta}-F_{\beta}^{2} = \gamma(F_{\beta}F_{\gamma\beta}-F_{\gamma}F_{\beta\beta}-\alpha F_{\gamma\beta}), \qquad (2.10)$$

and a simple similarity solution arises if it is assumed that F is independent of γ, i.e.

$$F = F_0(\beta),$$

so that

$$u = xt^{-1}F_0'(\beta).$$

Hoult suggests that this is an appropriate description of the flow throughout the boundary layer, but this cannot be correct. The basic objection is that the boundary-layer thickness, in such a description, is independent of x, and this must be incorrect in the vicinity of the leading edge. A more general structure, capable of meeting this objection, is

$$F(\gamma,\beta) = F_0(\beta)+\sum_{n=1}^{\infty} C_n\gamma^{\omega_n}F_n(\beta), \qquad (2.11)$$

where $\{\omega_n\}$ is a positive increasing sequence of real numbers. The equations satisfied by F_0 and F_1 are

$$F_0'''+F_0''(\tfrac{1}{2}\beta+F_0)+F_0'-F_0'^{2} = 0, \qquad (2.12)$$

$$F_1'''+F_1''(\tfrac{1}{2}\beta+F_0)+F_1'(1+\tfrac{3}{8}\omega_1-2F_0'-\omega_1F_0')+F_1(1+\omega_1)F_0'' = 0 \qquad (2.13)$$

and the solutions of these equations have been investigated numerically. In order to do this it is necessary to specify certain boundary conditions at the wall. From (2.6*a*),

$$F(\gamma,0) = 0$$

is immediate, but since q is unknown, the best we can do is represent it as a series in powers of γ. This introduces too much generality to make a numerical investigation of (2.12) and (2.13) convenient, however, so that instead we shall take

$$F_\beta(\gamma,0) = \alpha,$$

corresponding to

$$q = \alpha xt^{-1}.$$

The broad qualitative features of solutions corresponding to this choice are unlikely to differ from those arising from a more general choice. Moreover, as will be shown later, this particular choice of q is the right one for our problem. The conditions on the F_n are then

$$F_n(0) = 0, \quad F_0'(0) = \alpha, \quad F_n'(0) = 0 \quad (n \geqslant 1), \quad \lim_{\beta\to\infty} F_n'(\beta) = 0. \qquad (2.14)$$

Unfortunately, these do not ensure a unique solution for the F_n. The essential difficulty can be isolated by considering the behaviour for large β. Linearization about the solution at infinity (F_n = constant) suggests that the asymptotic solution for F_n' contains two independent terms, both of which vanish in the limit, namely

$$\beta^{-2-\frac{3}{4}\omega_n}, \quad \beta^{1+\frac{3}{4}\omega_n}\exp\{-\tfrac{1}{4}[\beta+2F_n(\infty)]^2\}.$$

Consequently, *any* choice of $F_n''(0)$ will lead to a solution satisfying the given boundary conditions.

The nature of the solutions for F_0 has been investigated numerically, for a choice of α equal to $\frac{3}{8}$.

If $F_0''(0)$ is positive, there are points in the boundary layer at which the water velocity exceeds that of the slick at the same values of x and t. Intuitively, we might feel justified in excluding such solutions.

If $F_0''(0) < -0{\cdot}0124$ the boundary layer contains regions of reversed flow, and presumably these also can be rejected on intuitive grounds.†

For values of $F_0''(0)$ in the closed interval $[-0{\cdot}0124, 0]$, the x component of velocity is a monotone decreasing function of β. Moreover, at the left-hand limit of the interval the behaviour at infinity is exponential, whereas for all other points it is algebraic. It may be conjectured, then, that

$$F_0''(0) = -0{\cdot}0124. \tag{2.15}$$

The reasons for favouring the exponential solution have their origin in higher order boundary-layer theory (see the discussion by Brown & Stewartson (1965)). More precisely, it may be argued that the boundary-layer flow must approach the free-stream conditions exponentially, except possibly at isolated points. Consequently, an expansion of the form (2.11), in which each term has algebraic decay, cannot be valid for large β. If, as sometimes is the case, exponential behaviour cannot be forced, an 'outer' expansion must be found in which each term exhibits exponential decay except at $x = 0$ (the acceptable isolated point). This outer expansion must match, in the usual sense, with the 'inner' expansion (2.11). In the present problem, exponential decay *can* be forced for the inner expansion, and doing this is tantamount to assuming that an alternative, two-layer structure does not exist. It boils down, in the end, to a uniqueness assumption.

The function F_0 is now uniquely determined, and we may turn to the equation for F_1. Since the equation and its boundary conditions are homogeneous we may choose $F_1''(0) = 1$ without loss of generality. Exponential behaviour is then forced by appropriate choice of the parameter ω_1. Indeed it can be anticipated that there is a countably infinite number of possible values of ω_1 which will lead to an acceptable solution. Numerical integration shows that the first two in this list are

$$4{\cdot}966, \quad 8{\cdot}248. \tag{2.16}$$

Such large values imply that the skin friction varys most rapidly in the vicinity of the leading edge ($\gamma = 1$).

In principle, all the F_n and ω_n may now be found, in a systematic manner, but this does not lead to a uniquely defined boundary-layer flow. The reason is that an infinite number of the C_n in (2.11) are undetermined. Such non-uniqueness is typical of downstream expansions of *steady* boundary-layer flows; it arises in these other problems because of failure to account adequately for initial conditions (upstream data). In all known cases for which the boundary layer is viscous, the non-uniqueness manifests itself in the form of an infinite

† Hoult's (1972) solution exhibits reversed flow but this is a direct result of insisting that the one-term similarity solution is valid all the way to the leading edge.

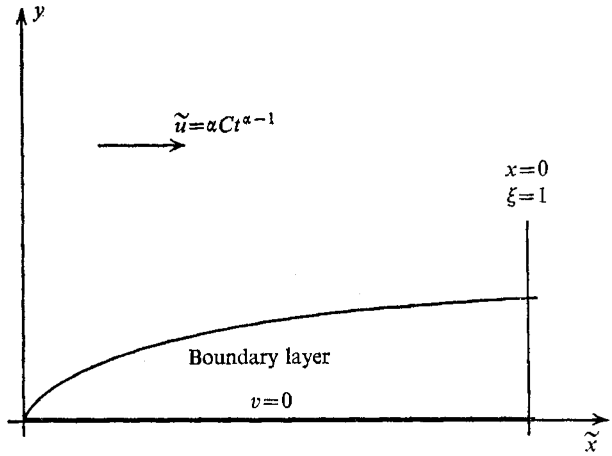

FIGURE 2. Boundary layer in leading-edge-fixed frame.

number of undetermined constants. (Grosser manifestations are possible if the boundary layer contains an inviscid region, and this is presumably related to the downstream convection of an unknown distribution of vorticity. Examples are given in the work of Buckmaster (1971) and Burggraf, Stewartson & Belcher (1971).) It is reasonable to conclude, then, that in some sense the expansion (2.11) is a 'downstream' expansion. Thus it might be more fruitful to integrate from the leading edge of the slick.

2.2. *Structure in the neighbourhood of the leading edge*

The solution near the leading edge of the slick is best described in a frame that is moving with the right-hand edge of the slick (figure 2). In this frame, the boundary-layer equations become

$$\left.\begin{aligned}\frac{\partial\tilde{u}}{\partial t}+\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}}+v\frac{\partial\tilde{u}}{\partial y} &= \ddot{R}+\nu\frac{\partial^2\tilde{u}}{\partial y^2},\\ \partial\tilde{u}/\partial\tilde{x}+\partial v/\partial y &= 0.\end{aligned}\right\} \tag{2.17}$$

where $\tilde{x} = R(t)-x, \quad \tilde{u} = \dot{R}(t)-u.$

The boundary conditions are

$$v(\tilde{x},0;t) = 0,$$

$$\lim_{y\to\infty}\tilde{u}(\tilde{x},y;t) = \alpha Ct^{\alpha-1}, \quad \lim_{\tilde{x}\to 0}\tilde{u}(\tilde{x},0;t) = 0.$$

Introducing the stream function by

$$\tilde{u} = \partial\tilde{\psi}/\partial y, \quad v = -\partial\tilde{\psi}/\partial\tilde{x},$$

we seek a solution in the form

$$\tilde{\psi} = (2\alpha C\nu\tilde{x}t^{\alpha-1})^{\frac{1}{2}}\,G(\xi,\eta), \tag{2.18}$$

where $\xi = \tilde{x}/Ct^{\alpha}, \quad \eta = y(\alpha Ct^{\alpha-1}/2\nu\tilde{x})^{\frac{1}{2}}.$

The function G satisfies the equation

$$G_{\eta\eta\eta}+GG_{\eta\eta} = 2\xi(G_\eta G_{\eta\xi}-G_\xi G_{\eta\eta}-\xi G_{\eta\xi})+2\xi(\alpha-1)\alpha^{-1}(G_\eta+\tfrac{1}{2}\eta G_{\eta\eta}-1) \quad (2.19)$$

together with the boundary conditions

$$G(\xi,0)=0, \quad \lim_{\eta\to\infty} G_\eta(\xi,\eta)=1.$$

Expansion of G as a power series in ξ,

$$G(\xi,\eta)=\sum_{n=0}^{\infty}\xi^n G_n(\eta),$$

then leads to a sequence of equations satisfied by the G_n, the first few of which are

$$G_0'''+G_0G_0''=0, \quad (2.20a)$$

$$G_1''''+G_0G_1''-2G_0'G_1'+3G_0''G_1=\tfrac{10}{3}-\tfrac{10}{3}(G_0'+\tfrac{1}{2}\eta G_0''), \quad (2.20b)$$

$$G_2''''+G_0G_2''-4G_0'G_2'+5G_0''G_2=2G_1'^2-3G_1G_1''-2G_1'-\tfrac{10}{3}(G_1'+\tfrac{1}{2}\eta G_1''), \quad (2.20c)$$

$$G_3''''+G_0G_3''-6G_0'G_3'+7G_0''G_3=6G_1'G_2'-3G_1G_2''-5G_2G_1'' -4G_2'-\tfrac{10}{3}(G_2'+\tfrac{1}{2}\eta G_2''). \quad (2.20d)$$

These are to be solved subject to the boundary conditions

$$G_n(0)=0, \quad \lim_{\eta\to\infty} G_n'(\eta)=\delta_{n0}, \quad G_0'(0)=0. \quad (2.21)$$

Note that, for the moment, the $G_n'(0)$ are not known for $n \geqslant 1$. Nor, for that matter, are the $G_n''(0)$, but they are related to the $G_n'(0)$ through equations (2.20).

The skin friction at the oil–water interface is

$$\frac{\partial \tilde{u}}{\partial y}(\tilde{x},0;t)=(\alpha Ct^{\alpha-1})^{\frac{3}{2}}(2\nu\tilde{x})^{-\frac{1}{2}}\sum_{n=0}^{\infty}\xi^n G_n''(0)$$

and using this it is possible to write down an expansion for the thickness of the slick. Specifically, we integrate (2.5) and find that

$$h^2=(2\nu)^{\frac{1}{2}}(\alpha C)^{\frac{3}{2}}(g\Delta)^{-1}\tilde{x}^{\frac{1}{2}}t^{\frac{3}{2}(\alpha-1)}\sum_{n=0}^{\infty}\xi^n\frac{G_n''(0)}{n+\frac{1}{2}}. \quad (2.22)$$

At this juncture, the value of α may be determined by consideration of the volume V of the slick. This is defined as

$$V=Ct^\alpha\int_0^1 h\,d\xi.$$

It may be objected that this is, in reality, but one half of the volume. However, our definition agrees with that of Hoult (1972) and we retain it to facilitate comparison with experimental results cited in his article.

With the aid of (2.22), the volume can be written as

$$V=C^2\alpha^{\frac{3}{4}}(2\nu)^{\frac{1}{4}}(g\Delta)^{-\frac{1}{2}}t^{2\alpha-\frac{3}{4}}\int_0^1 d\xi\,\xi^{\frac{1}{4}}\left[\sum_{n=0}^{\infty}\frac{\xi^n G_n''(0)}{n+\frac{1}{2}}\right]^{\frac{1}{2}}, \quad (2.23)$$

and this is only independent of time if α is assigned the value $\frac{3}{8}$. An alternative problem, which might have practical significance, corresponds to a choice of V

that increases linearly with time (source problem). This corresponds to a value of α equal to $\frac{7}{8}$ and we do not consider it here.

The speed of the oil slick is given by the expression

$$q = \alpha C t^{\alpha-1} - \alpha C t^{\alpha-1} \sum_{n=0}^{\infty} \xi^n G_n'(0). \tag{2.24}$$

Substituting this and (2.22) into the oil continuity equation (2.4), which may be written in the form

$$\frac{\partial}{\partial t}(h^2) + \alpha C t^{\alpha-1} \frac{\partial}{\partial \tilde{x}}(h^2) - q \frac{\partial}{\partial \tilde{x}}(h^2) - 2h^2 \frac{\partial q}{\partial \tilde{x}} = 0,$$

yields a relation between the components of q and the components of the skin friction, namely

$$\sum_{n=0}^{\infty} \xi^{n+1} \frac{G_n''(0)}{n+\frac{1}{2}} \left[\frac{3}{2}\frac{(\alpha-1)}{\alpha} - n\right] + \sum_{n=0}^{\infty} \xi^n G_n'(0) \sum_{n=0}^{\infty} \xi^n G_n''(0)$$
$$+ 2 \sum_{n=0}^{\infty} \frac{\xi^n G_n''(0)}{n+\frac{1}{2}} \sum_{n=0}^{\infty} \xi^n n G_n'(0) = 0. \tag{2.25}$$

It is easily verified that, when $\alpha = \frac{3}{8}$, the solution of this infinite set of equations is

$$G_1'(0) = 1, \quad G_n'(0) = 0, \quad n \geqslant 2. \tag{2.26}$$

Remarkably enough, the velocity of the oil is then given by the very simple expression

$$q = \tfrac{3}{8} x/t \tag{2.27}$$

as was assumed in § 2.1. An alternative way of deducing this result is to note that, when $\alpha = \frac{3}{8}$, the continuity equation is identically satisfied by solutions of the form

$$q = \alpha x/t, \quad h = t^{\alpha-\frac{3}{4}} f(\xi),$$

where f is an arbitrary function. No such simple solution exists for different choices of α (e.g. the source problem) and for the remainder of the paper α will be explicitly assigned the value $\frac{3}{8}$.

With the velocity components at the interface now specified, the G_n are uniquely defined, and in particular the $G_n''(0)$ may be systematically evaluated by numerical means. The thickness of the slick is now known in principle since it is given by (2.22), and moreover C may be determined from (2.23) by summing the series.

Table 1 shows the first six skin-friction coefficients. Unfortunately, these are too large to lend any credence to the idea that the series can be easily summed. This is consistent with the prediction of § 2.1 that the skin friction varies rapidly in the vicinity of the leading edge, and forces us to construct the solution in a different manner. This is the subject of § 3.

n	0	1	2	3	4	5
$G_n''(0)$	0·46960	−1·99999	1·19382	2·09429	4·78127	8·71400

TABLE 1

3. Evaluation of the slick size

The essence of the problem is to solve the equations

$$\frac{\partial \tilde{u}}{\partial t}+\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}+v\frac{\partial \tilde{u}}{\partial y} = -\tfrac{15}{64}Ct^{-\frac{13}{8}}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{3.1a}$$

$$\partial\tilde{u}/\partial\tilde{x}+\partial v/\partial y = 0 \tag{3.1b}$$

subject to the boundary conditions

$$\tilde{u}(\tilde{x},0;t) = \tfrac{3}{8}Ct^{-\frac{5}{8}}\xi, \quad v(\tilde{x},0;t) = 0,$$

$$\lim_{y\to\infty}\tilde{u}(\tilde{x},y;t) = \tfrac{3}{8}Ct^{-\frac{5}{8}}.$$

Note that this complete formulation is only possible because of the results of § 2, specifically (2.27).

This problem is readily solved numerically after being cast in terms of the variables of § 2.2. Specifically, (2.19) may be integrated from the leading edge, using a step-by-step procedure. The solution is started using the series expansion for small ξ developed in § 2.2 and then is continued using, essentially, the method described by Terrill (1960). The thickness of the boundary layer on the η scale is found to decrease with increasing ξ.

The skin friction is of primary interest and was computed at intervals in ξ of 0·025 starting at $\xi = 0{\cdot}1$, the point where the series expansion (the six leading terms) was abandoned. Table 2 shows some of the results, including those of the series expansion.

The results in the vicinity of $\xi = 1$ may be compared with the local solution described in § 2.1. We find that $G_{\eta\eta}(0{\cdot}9, 0)$, evaluated numerically, differs from the prediction of the one-term similarity solution by about 2×10^{-4}. This must represent the limit in accuracy of both calculations.

Once the skin friction is known, the slick thickness may be found (cf. equation (2.22)). Table 3 shows values of $H(\xi)$, where

$$h \equiv \nu^{\frac{1}{4}}(g\Delta)^{-\frac{1}{2}}Ct^{-\frac{3}{8}}H(\xi). \tag{3.2}$$

It is apparent that the slick is very blunt in shape, coming within 10 % of its maximum thickness at a value of ξ less than 0·2.

The calculation is completed by integrating the thickness to find the volume (cf. equation (2.23)). This leads to a value of C, whence the size of the slick is

$$R(t) = 1{\cdot}76(g\Delta)^{\frac{1}{4}}V^{\frac{1}{2}}\nu^{-\frac{1}{8}}t^{\frac{3}{8}}. \tag{3.3}$$

This is to be compared with experimental results, for which the coefficient that best fits the data is 1·5 (Hoult 1972). Thus the error is approximately 17 %.

ξ	$G_{\eta\eta}(\xi, 0)$ Numerical	$G_{\eta\eta}(\xi, 0)$ Series	ξ	$G_{\eta\eta}(\xi, 0)$ Numerical	$G_{\eta\eta}(\xi, 0)$ Series
0	—	0·4696	0·275	0·084	0·095
0·025	—	0·420	0·30	0·072	0·093
0·05	—	0·373	0·325	0·062	0·103
0·075	—	0·327	0·35	0·055	—
0·10	—	0·284	0·40	0·045	—
0·125	0·244	0·244	0·50	0·036	—
0·150	0·207	0·207	0·60	0·028	—
0·175	0·173	0·173	0·70	0·021	—
0·20	0·144	0·145	0·80	0·015	—
0·225	0·119	0·121	0·90	0·008	—
0·25	0·099	0·104			

TABLE 2

ξ	$H(\xi)$	ξ	$H(\xi)$
0	0	0·15	0·308
0·002	0·117	0·20	0·319
0·004	0·138	0·30	0·329
0·006	0·153	0·40	0·334
0·008	0·164	0·50	0·337
0·01	0·173	0·60	0·339
0·02	0·205	0·70	0·340
0·03	0·225	0·80	0·341
0·05	0·252	0·90	0·342
0·075	0·274	1·0	0·342
0·10	0·289		

TABLE 3

Although the theoretical result is in substantial agreement with experiment, the error is, perhaps, larger than might have been anticipated *a priori*. After all, it would be necessary to *double* the theoretical skin friction to get the coefficient within a few per cent of 1·5. The reason for the discrepancy is not clear. Perhaps the boundary layer is turbulent, rather than laminar, as assumed here.

REFERENCES

BROWN, S. N. & STEWARTSON, K. 1965 *J. Fluid Mech.* **23**, 673.
BUCKMASTER, J. D. 1971 *Quart. J. Mech. Appl. Math.* **24**, 373.
BURGGRAF, O. R., STEWARTSON, K. & BELCHER, R. 1971 *Phys. Fluids*, **14**, 1821.
FAY, J. A. 1969 In *Oil on the Sea* (ed. D. P. Hoult), pp. 53–63. Plenum.
HOULT, D. P. 1972 *Ann. Rev. Fluid Mech.* **4**, 341.
TERRILL, R. M. 1960 *Phil. Trans.* A **253**, 55.